

Isomorphisms of non noetherian down-up algebras

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Abstract

We solve the isomorphism problem for non noetherian down-up algebras $A(\alpha, 0, \gamma)$ by lifting isomorphisms between some of their non commutative quotients. The quotients we consider are either quantum polynomial algebras in two variables for $\gamma = 0$ or quantum versions of the Weyl algebra A_1 for non zero γ . In particular we obtain that no other down-up algebra is isomorphic to the monomial algebra $A(0, 0, 0)$. We prove in the second part of the article that this is the only monomial algebra within the family of down-up algebras. Our method uses homological invariants that determine the shape of the possible quivers and we apply the abelianization functor to complete the proof.

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1 Introduction

Let \mathbb{k} be a fixed field of characteristic 0. Given parameters $(\alpha, \beta, \gamma) \in \mathbb{k}^3$, the associated *down-up algebra* $A(\alpha, \beta, \gamma)$, first defined in [2], is the quotient of the free associative algebra $\mathbb{k}\langle d, u \rangle$ by the ideal generated by the relations

$$\begin{aligned} d^2u - (\alpha dud + \beta ud^2 + \gamma d), \\ du^2 - (\alpha udu + \beta u^2d + \gamma u). \end{aligned}$$

There are several well-known examples of down-up algebras such as $A(2, -1, 0)$, isomorphic to the enveloping algebra of the Heisenberg-Lie algebra of dimension 3, and, for $\gamma \neq 0$, $A(2, -1, \gamma)$, isomorphic to the enveloping algebra of $\mathfrak{sl}(2, \mathbb{k})$.

A down-up algebra has a PBW basis given by

$$\{u^i(du)^j d^k : i, j, k \geq 0\}.$$

Note that $A(\alpha, \beta, \gamma)$ can be regarded as a \mathbb{Z} -graded algebra where the degrees of u and d are respectively 1 and -1 . The field \mathbb{k} is the trivial module over $A(\alpha, \beta, \gamma)$, with d and u acting as 0.

E. Kirkman, I. Musson and D. Passman proved in [7] that $A(\alpha, \beta, \gamma)$ is noetherian if and only if it is a domain, if and only if $\beta \neq 0$.

The isomorphism problem for down-up algebras was posed in [2] where the authors considered algebras $A(\alpha, \beta, \gamma)$ of four different types and proved, by studying one dimensional modules, that algebras of different types are not isomorphic. They considered the following types,

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- (a) $\gamma = 0, \alpha + \beta = 1,$ (c) $\gamma \neq 0, \alpha + \beta = 1,$
(b) $\gamma = 0, \alpha + \beta \neq 1,$ (d) $\gamma \neq 0, \alpha + \beta \neq 1.$

As a consequence, we can restrict the question of whether two down-up algebras are isomorphic to each of the four types. In [3] the authors solved the isomorphism problem for noetherian down-up algebras of types (a), (b) and (c) for every algebraically closed field \mathbb{k} , and also for noetherian algebras of type (d) when in addition $\text{char}(\mathbb{k}) = 0$. More precisely, they proved the following result.

Theorem ([3]). *Let $A = A(\alpha, \beta, \gamma)$ and $A' = A(\alpha', \beta', \gamma')$ be noetherian down-up algebras. Then A is isomorphic to A' if and only if*

1. $\gamma = \lambda\gamma'$ for some $\lambda \in \mathbb{k}^\times$, and
2. either $\alpha' = \alpha, \beta' = \beta$ or $\alpha' = -\alpha\beta^{-1}, \beta' = \beta^{-1}$.

Their solution focuses mainly on the possible commutative quotients of down-up algebras. In contrast with this, there are very well studied non commutative algebras that appear as quotients of non noetherian down-up algebras, for example, when $\alpha \in \mathbb{k}^\times$, the quantum plane $\mathbb{k}_\alpha[x, y]$ and the quantum Weyl algebra A_α^1 ,

$$\mathbb{k}_\alpha[x, y] := \mathbb{k}\langle x, y \rangle / (yx - \alpha xy), \quad A_\alpha^1 := \mathbb{k}\langle x, y \rangle / (yx - \alpha xy - 1).$$

In this article we describe isomorphisms amongst non noetherian down-up algebras by using these quotients. Our main result is the following.

Theorem 1.1. *Let \mathbb{k} be an algebraically closed field and let $\alpha, \alpha', \gamma, \gamma' \in \mathbb{k}$. The algebras $A(\alpha, 0, \gamma)$ and $A(\alpha', 0, \gamma')$ are isomorphic if and only if*

1. $\gamma = \lambda\gamma'$, for some $\lambda \in \mathbb{k}^\times$, and
2. $\alpha' = \alpha$.

We obtain in particular that no other down-up algebra is isomorphic to $A(0, 0, 0)$. In Section 3 we prove

Theorem 1.2. *The algebra $A(\alpha, \beta, \gamma)$ is monomial if and only if $(\alpha, \beta, \gamma) = (0, 0, 0)$.*

So, the only monomial algebra in the family of down-up algebras is the evident one. Our starting point is the fact that noetherian down-up algebras cannot be monomial since they are a domain of global dimension 3 [7]. The situation can be related to 3-dimensional Sklyanin algebras. In both cases, an algebra A is noetherian if and only if it is a domain. For Sklyanin algebras, these conditions are equivalent to A being monomial [9]. This is not the case for down-up algebras.

Our proof uses homological invariants that determine the possible shapes of the quiver. We think that these methods may be useful for other families of algebras.

2 Isomorphisms of non noetherian down-up algebras

The purpose of this section is to prove Theorem 1.1. Let \mathbb{k} be an algebraically closed field. Note that the condition $\gamma = \lambda\gamma'$ for $\lambda \in \mathbb{k}^\times$ is equivalent to the condition of γ and γ' being both zero or both non zero. We already know from [2] that if $\gamma \neq 0$, then $A(\alpha, 0, \gamma)$ is isomorphic to $A(\alpha, 0, 1)$. This is done by rescaling d by γd . Also, observe that $A(\alpha, 0, 0)$ is not isomorphic to $A(\alpha', 0, 1)$ for all $\alpha, \alpha' \in \mathbb{k}$, since they belong to different types. Gathering all this information, we deduce that Theorem 1.1 is equivalent to the following two propositions:

Proposition 2.1. *Let $\alpha, \alpha' \in \mathbb{k}$. The algebras $A(\alpha, 0, 0)$ and $A(\alpha', 0, 0)$ are isomorphic if and only if $\alpha = \alpha'$.*

Proposition 2.2. *Let $\alpha, \alpha' \in \mathbb{k}$. The algebras $A(\alpha, 0, 1)$ and $A(\alpha', 0, 1)$ are isomorphic if and only if $\alpha = \alpha'$.*

We will thus prove both of them in order to obtain our result.

Lemma 2.3. *Let $\alpha \in \mathbb{k}^\times$, $\gamma \in \mathbb{k}$ and let $A = A(\alpha, 0, \gamma)$ be a down-up algebra. Denote $\omega := du - \alpha ud - \gamma$. The algebra $A/\langle \omega \rangle$ is isomorphic to $\mathbb{k}_\alpha[x, y]$ if $\gamma = 0$ and it is isomorphic to A_α^1 if $\gamma \neq 0$. Moreover, in case $\gamma = 0$ or $\gamma = 1$, the isomorphism maps the class of d to y and the class of u to x .*

Proof. The algebra $A(\alpha, 0, \gamma)$ is the quotient of the free algebra generated by the variables d and u subject to the relations $d^2u - \alpha dud - \gamma d = 0$ and $du^2 - \alpha udu - \gamma u = 0$. Denote by Ω the element $du - \alpha ud - \gamma$ in the free algebra. The projection of Ω onto $A(\alpha, 0, \gamma)$ is ω . The defining relations of A are $d\Omega = 0$ and $\Omega u = 0$. Therefore, the algebra $A/\langle \omega \rangle$ is isomorphic to the algebra freely generated by letters d, u subject to the relation $\Omega = 0$. If $\gamma = 0$, then this is exactly the definition of $\mathbb{k}_\alpha[x, y]$. If $\gamma \neq 0$, then $\omega = \gamma^{-1}((\gamma d)u - \alpha u(\gamma d) - 1)$, and so $A/\langle \omega \rangle$ is the quantum Weyl algebra generated by x and y , with $y = \gamma d$ and $x = u$. \square

In [8] the authors describe all isomorphisms and automorphisms for quantum Weyl algebras A_α^1 for $\alpha \in \mathbb{k}^\times$ not a root of unity. In [10] this result is generalized to the family of *quantum generalized Weyl algebras*, including the quantum plane and the quantum Weyl algebra for all values of $\alpha \in \mathbb{k}^\times$. We recall some of their results in the cases relevant to us.

Theorem 2.4 ([8],[10]). *Let $\alpha, \alpha' \in \mathbb{k} \setminus \{0, 1\}$.*

- i) *The algebras $\mathbb{k}_\alpha[x, y]$ and $\mathbb{k}_{\alpha'}[x, y]$ are isomorphic if and only if $\alpha' \in \{\alpha, \alpha^{-1}\}$. Moreover, if $\varphi : \mathbb{k}_\alpha[x, y] \rightarrow \mathbb{k}_{\alpha^{-1}}[x, y]$ is an isomorphism and $\alpha \neq -1$, then there exist $\lambda, \mu \in \mathbb{k}^\times$ such that $\varphi(x) = \lambda y$ and $\varphi(y) = \mu x$.*
- ii) *The algebras A_α^1 and $A_{\alpha'}^1$ are isomorphic if and only if $\alpha' \in \{\alpha, \alpha^{-1}\}$. If $\alpha \neq -1$, then every isomorphism $\eta : A_\alpha^1 \rightarrow A_{\alpha^{-1}}^1$ is of the form $\eta(x) = \lambda y$ and $\eta(y) = -\lambda^{-1}\alpha^{-1}x$, for some $\lambda \in \mathbb{k}^\times$.*

Proof of Proposition 2.1. Let α and α' be elements of \mathbb{k} and suppose there exists an isomorphism of \mathbb{k} -algebras $\varphi : A(\alpha, 0, 0) \rightarrow A(\alpha', 0, 0)$. Denote $A := A(\alpha, 0, 0)$, $A' := A(\alpha', 0, 0)$ and let d' and u' be the usual generators of A' .

Suppose $\alpha, \alpha' \in \mathbb{k} \setminus \{0, 1\}$ and $\alpha \neq \alpha'$. Let $\omega := du - \alpha ud$ and $\omega' := d'u' - \alpha' u'd'$. By Lemma 2.3, we can identify $A/\langle \omega \rangle$ with $\mathbb{k}_\alpha[x, y]$, where the canonical projection $\pi : A \rightarrow \mathbb{k}_\alpha[x, y]$ sends d to y and u to x , and similarly for $A'/\langle \omega' \rangle$ and $\mathbb{k}_{\alpha'}[x, y]$; here we denote by π' the canonical projection. Define $\psi_1 = \pi' \circ \varphi$. The equalities $d\omega = 0$ and $\omega u = 0$ hold in A , so

$$\psi_1(d)\psi_1(\omega) = 0 = \psi_1(\omega)\psi_1(u).$$

The algebra $\mathbb{k}_{\alpha'}[x, y]$ is a non commutative domain generated by $\psi_1(d)$ and $\psi_1(u)$. Thus, $\psi_1(d)$ and $\psi_1(u)$ are not zero and from the above equations we deduce $\psi_1(\omega) = 0$. This implies that there exists an algebra map $\bar{\psi}_1 : \mathbb{k}_\alpha[x, y] \rightarrow \mathbb{k}_{\alpha'}[x, y]$ such that $\psi_1 = \bar{\psi}_1 \circ \pi$. In the other direction we obtain that $\psi_2 := \pi \circ \varphi^{-1}$ factors as $\psi_2 = \bar{\psi}_2 \circ \pi'$. Since $\bar{\psi}_1 \circ \bar{\psi}_2 \circ \pi' = \pi'$ and $\bar{\psi}_2 \circ \bar{\psi}_1 \circ \pi = \pi$, we deduce $\bar{\psi}_1$ is an

isomorphism. The situation is illustrated by the following commutative diagram,

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A' \\ \pi \downarrow & & \downarrow \pi' \\ \mathbb{k}_\alpha[x, y] & \xrightleftharpoons[\psi_2]{\bar{\psi}_1} & \mathbb{k}_{\alpha'}[x, y] \end{array}$$

By Theorem 2.4 and our assumption that $\alpha \neq \alpha'$, we obtain $\alpha' = \alpha^{-1}$. Theorem 2.4 also says that there exist $\lambda, \mu \in \mathbb{k}^\times$ and $z_1, z_2 \in \langle \omega' \rangle$ such that $\varphi(u) = \lambda d' + z_1$ and $\varphi(d) = \mu u' + z_2$. Note that A' is graded considering the generators d' and u' in degree 1. Since $\deg(\omega') = 2$, it follows that z_1 and z_2 are either 0 or sums of homogeneous elements of degree at least 2 with respect to this grading. On the other hand $d^2u - \alpha dud$ is 0, and so

$$0 = \varphi(d)^2 \varphi(u) - \alpha \varphi(d) \varphi(u) \varphi(d).$$

In particular the degree 3 component of the right hand side of this equality, that is $(u')^2 d' - \alpha u' d' u'$, must be 0. But the set $\{(u')^i (d' u')^j (d')^l : i, j, l \in \mathbb{N}_0\}$ is a \mathbb{k} -basis of A' , and this is a contradiction.

In case $\alpha = 0$, an argument similar to the above one shows that there is an epimorphism $\psi : A \rightarrow \mathbb{k}_{\alpha'}[x, y]$. As a consequence the elements $\psi(d)$ and $\psi(u)$ generate $\mathbb{k}_{\alpha'}[x, y]$. If $\alpha' \neq 0$, then the algebra $\mathbb{k}_{\alpha'}[x, y]$ is a domain and it is not commutative, thus it cannot be generated by one element. From the equality $0 = d^2u$ we obtain that $0 = \psi(d^2u) = \psi(d)^2 \psi(u)$, implying $\psi(d) = 0$ or $\psi(u) = 0$. This is a contradiction and so $\alpha' = 0$.

If $\alpha = 1$, then A belongs to type (a) and so does A' . This implies $\alpha' = 1$, concluding the proof of the proposition. \square

Now we turn our attention to Proposition 2.2. Let $A = A(\alpha, 0, 1)$ for $\alpha \in \mathbb{k}$. Recall that $\omega := du - \alpha ud - 1$. Using Lemma 2.2 in [11], the set $\{u^i \omega^j d^l : i, j, l \geq 0\}$ is a \mathbb{k} -basis of A .

Lemma 2.5. *The set $\{u^i \omega^j d^l : i, l \geq 0 \text{ and } j \geq 1\}$ is a \mathbb{k} -linear basis of the two sided ideal $\langle \omega \rangle$, and, for each $n \in \mathbb{N}$, the set $\{u^i \omega^j d^l : i, l \geq 0 \text{ and } j \geq n\}$ is a \mathbb{k} -linear basis of $\langle \omega \rangle^n$.*

Proof. Every element of the form $u^i \omega^j d^l$ with $j \geq 1$ belongs to $\langle \omega \rangle$, so it only remains to prove that $\langle \omega \rangle$ is contained in the \mathbb{k} -vector space with basis the set $\{u^i \omega^j d^l : i, l \geq 0 \text{ and } j \geq 1\}$. Given $z \in \langle \omega \rangle$, write $z = \sum_{i,j,l} \lambda_{i,j,l} u^i \omega^j d^l$ with $i, j, l \geq 0$ and $\lambda_{i,j,l} \in \mathbb{k}$. By Lemma 2.3 we can identify $A/\langle \omega \rangle$ with A_α^1 , and the canonical projection $\pi : A \rightarrow A_\alpha^1$ sends u to x and d to y . The set $\{x^i y^l : i, l \geq 0\}$ is a basis of A_α^1 . From the equalities $\sum_{i,l} \lambda_{i,0,l} x^i y^l = \pi(z) = 0$, we deduce $\lambda_{i,0,l} = 0$ for all $i, l \geq 0$.

Taking into account the description we now have of $\langle \omega \rangle$, we see that the elements of $\langle \omega \rangle^2$ are linear combinations of monomials of type $u^i \omega^j d^l u^{i'} \omega^{j'} d^{l'}$, with $j, j' \geq 1$. Similarly, the elements of $\langle \omega \rangle^n$ are linear combinations of n -fold products of the same type. Therefore, to prove the second claim, it is sufficient to show that for every $r, s \geq 0$ there exist $\lambda_i \in \mathbb{k}$ such that $\omega d^r u^s \omega = \sum_{i \geq 2} \lambda_i \omega^i$. Indeed, there exist $\lambda_{i,j,l} \in \mathbb{k}$ such that

$$d^r u^s = \sum_{i,j,l \geq 0} \lambda_{i,j,l} u^i \omega^j d^l.$$

So

$$\omega d^r u^s \omega = \sum_{i,j,l \geq 0} \lambda_{i,j,l} \omega u^i \omega^j d^l \omega = \sum_{j \geq 0} \lambda_{0,j,0} \omega^{j+2}.$$

The last equality follows from $d\omega = 0$ and $\omega u = 0$. \square

Corollary 2.6. *The set $\{[u^i \omega d^l] : i, l \geq 0\}$, where $[p]$ denotes the class of an element p in $\langle \omega \rangle / \langle \omega \rangle^2$, is a \mathbb{k} -linear basis of the A -bimodule $\langle \omega \rangle / \langle \omega \rangle^2$. Moreover, in case $\alpha \neq 1$, the following equalities hold*

$$\begin{aligned} [u^i \omega d^l u] &= \frac{\alpha^l - 1}{\alpha - 1} [u^i \omega d^{l-1}], \\ [du^i \omega d^l] &= \frac{\alpha^i - 1}{\alpha - 1} [u^{i-1} \omega d^l], \end{aligned}$$

where the terms on the right are considered to be zero for $l = 0$ or $i = 0$.

Proof. The first claim is a direct consequence of Lemma 2.5. To prove the first formula, we fix $i \geq 0$ and proceed by induction on l , the case $l = 0$ being trivial from the equalities $\omega u = 0 = d\omega$. On the other hand, since $\omega^2 = \omega(du - \alpha ud - 1) = \omega du - \omega$, we obtain that $\omega du = \omega^2 + \omega$. Similarly $du\omega = \omega^2 + \omega$. Therefore, $[u^i \omega du] = [u^i \omega]$. Now, for $l \geq 2$

$$\begin{aligned} [u^i \omega d^l u] &= [u^i \omega d^{l-2}(\alpha dud + d)] = \alpha[u^i \omega d^{l-1}ud] + [u^i \omega d^{l-1}] \\ &= \alpha \frac{\alpha^{l-1} - 1}{\alpha - 1} [u^i \omega d^{l-2}d] + [u^i \omega d^{l-1}] \\ &= \frac{\alpha^l - 1}{\alpha - 1} [u^i \omega d^{l-1}]. \end{aligned}$$

The second formula can be proved analogously. \square

Proof of Proposition 2.2. Let $\alpha, \alpha' \in \mathbb{k}$. Denote $A := A(\alpha, 0, 1)$, $A' := A(\alpha', 0, 1)$. Let d', u' be the generators of A' . Suppose there exists an isomorphism of \mathbb{k} -algebras $\varphi : A \rightarrow A'$. Recall that $\omega = du - \alpha ud - 1$ and $\omega' = d'u' - \alpha' u'd' - 1$.

If $\alpha = 1$, then A belongs to type (a), and so does A' , hence $\alpha' = 1$. Now suppose $\alpha = 0$ and $\alpha' \neq 0$. By Lemma 2.3 the algebra $A' / \langle \omega' \rangle$ is isomorphic to $A_{\alpha'}^1$, and, if π' denotes the canonical projection, then $\pi'(u') = x$ and $\pi'(d') = y$. Let $\psi = \pi' \circ \varphi$. Since $d\omega = \omega u = 0$, we have $\psi(d)\psi(\omega) = \psi(\omega)\psi(u) = 0$. Note that $\psi(d)$ and $\psi(u)$ generate $A_{\alpha'}^1$, and therefore they cannot belong to \mathbb{k} ; in particular they cannot be zero. We deduce $0 = \psi(\omega) = \psi(d)\psi(u) - 1$. The algebra $A_{\alpha'}^1$ has a filtration whose associated graded algebra $\text{Gr}(A_{\alpha'}^1)$ is $\mathbb{k}_{\alpha'}[x, y]$. The equality $\psi(d)\psi(u) = 1$ implies that $\text{Gr}(A_{\alpha'}^1)$ is not a domain, which is a contradiction since $\alpha' \neq 0$.

Suppose $\alpha, \alpha' \in \mathbb{k} \setminus \{0, 1\}$ and $\alpha \neq \alpha'$. By the same arguments as in the proof of Proposition 2.1, the map $\psi := \pi' \circ \varphi : A \rightarrow A_{\alpha'}^1$ induces an isomorphism of \mathbb{k} -algebras $\bar{\psi} : A_{\alpha}^1 \rightarrow A_{\alpha'}^1$. Theorem 2.4 implies $\alpha' = \alpha$ or $\alpha' = \alpha^{-1}$. Since we are assuming $\alpha \neq \alpha'$, we deduce $\alpha' = \alpha^{-1}$. Again, by Theorem 2.4 we obtain that there exist $\lambda \in \mathbb{k}^\times$ and $z_1, z_2 \in \langle \omega \rangle$ such that

$$\begin{aligned} \varphi^{-1}(d') &= -\lambda^{-1}\alpha u + z_1, \\ \varphi^{-1}(u') &= \lambda d + z_2. \end{aligned}$$

By rescaling the variables d, u , we may assume $\lambda = 1$. The equality $(d')^2 u' - \alpha^{-1} d' u' d' - d' = 0$ implies

$$\begin{aligned} 0 &= \varphi^{-1}(d')^2 \varphi^{-1}(u') - \alpha^{-1} \varphi^{-1}(d') \varphi^{-1}(u') \varphi^{-1}(d') - \varphi^{-1}(d') \\ &= \alpha^2 u^2 d - \alpha u d u + \alpha u + \alpha^2 u^2 z_2 - \alpha u z_1 d - \alpha z_1 u d \\ &\quad + u d z_1 - \alpha u z_2 u + z_1 d u - z_1 + z \\ &= -\alpha u \omega + \alpha(\alpha u^2 z_2 - u z_1 d) + (u d z_1 - \alpha u z_2 u) + z_1 \omega + z \in \langle \omega \rangle, \end{aligned}$$

where z denotes the sum of all terms in which at least two factors z_1 or z_2 are involved. Note that $z_1\omega$ and z belong to $\langle\omega\rangle^2$. Taking classes modulo $\langle\omega\rangle^2$,

$$\alpha[u\omega] + \alpha([uz_1d] - \alpha[u^2z_2]) = [udz_1] - \alpha[uz_2u].$$

Write now $z_1 = \sum_{i,l \geq 0, j \geq 1} \lambda_{i,j,l} u^i \omega^j d^l$ and $z_2 = \sum_{i,l \geq 0, j \geq 1} \mu_{i,j,l} u^i \omega^j d^l$. Using the formulas of Corollary 2.6 we obtain

$$\begin{aligned} \alpha[u\omega] + \sum_{i,l \geq 0} \alpha(\lambda_{i,1,l} [u^{i+1}\omega d^{l+1}] - \alpha\mu_{i,1,l} [u^{i+2}\omega d^l]) &= \\ &= \sum_{i \geq 1, l \geq 0} \lambda_{i,1,l} \frac{\alpha^i - 1}{\alpha - 1} [u^i \omega d^l] - \sum_{i \geq 0, l \geq 1} \alpha\mu_{i,1,l} \frac{\alpha^l - 1}{\alpha - 1} [u^{i+1} \omega d^{l-1}]. \end{aligned}$$

By Corollary 2.6, the set $\{[u^i \omega d^l] : i, l \geq 0\}$ is a \mathbb{k} -linear basis of $\langle\omega\rangle/\langle\omega\rangle^2$. For $m \geq 0$, define $\Lambda_m := \lambda_{m+1,1,m} - \alpha\mu_{m,1,m+1}$. Looking at the coefficient corresponding to the term $[u^{m+1} \omega d^m]$ in the last equation for each $m \geq 0$, we deduce

$$\begin{aligned} \alpha &= \Lambda_0, \\ \alpha\Lambda_{m-1} &= \frac{\alpha^{m+1} - 1}{\alpha - 1} \Lambda_m, \text{ for } m \geq 1. \end{aligned}$$

The fact that $\Lambda_0 = \alpha \neq 0$ implies, by an inductive argument, that $\Lambda_m \neq 0$ for all $m \in \mathbb{N}$. As a consequence, either $\lambda_{m+1,1,m} \neq 0$ for infinitely many values of $m \in \mathbb{N}$, or $\mu_{m,1,m+1} \neq 0$ for infinitely many values of $m \in \mathbb{N}$. This is a contradiction that comes from the assumption $\alpha \neq \alpha'$. \square

3 Monomial down-up algebras

An algebra is monomial if it is isomorphic to an algebra of the form $\mathbb{k}Q/I$, where Q is a quiver with a finite number of vertices and I is a two-sided ideal in $\mathbb{k}Q$ generated by paths of length at least 2. The algebra $A(0,0,0)$ is monomial and no other down-up algebra is isomorphic to it. However, other monomial down-up algebras may exist. In this section we prove Theorem 1.2. Before doing it we prove a series of preparatory lemmas.

We will make use of the abelianization functor defined on \mathbb{k} -algebras as

$$A \mapsto A^{\text{ab}} := A/J_A,$$

where J_A is the two sided ideal in A generated by the set $\{xy - yx : x, y \in A\}$. The canonical projection $\pi_A : A \rightarrow A^{\text{ab}}$ is a natural transformation from the identity to the abelianization functor.

In order to state the next lemma we need some previous definitions. Given a quiver Q with a finite number of vertices and $e, e' \in Q_0$, define $eQ_1e' := \{\alpha \in Q_1 : t(\alpha) = e, s(\alpha) = e'\}$, where t and s are the usual target and source maps. Also, denote by B_e the \mathbb{k} -algebra $\mathbb{k}[X_\alpha : \alpha \in eQ_1e]$. That is, B_e is the polynomial algebra in variables indexed by the elements of the set eQ_1e . In case $eQ_1e = \emptyset$ we set $B_e = \mathbb{k}$. If I is a two-sided ideal in $\mathbb{k}Q$ generated by paths of length at least 2, define I_e to be the ideal in B_e generated by the set

$$\bigcup_{n \geq 2} \{X_{\alpha_n} \cdots X_{\alpha_1} : \alpha_n \cdots \alpha_1 \in I, \alpha_i \in eQ_1e\}.$$

Lemma 3.1. *Let Q be a quiver with a finite number of vertices and I a two-sided ideal in $\mathbb{k}Q$ generated by paths of length at least 2. There is an isomorphism of \mathbb{k} -algebras*

$$\left(\frac{\mathbb{k}Q}{I}\right)^{\text{ab}} \cong \bigoplus_{e \in Q_0} \frac{B_e}{I_e}.$$

Proof. The classes \bar{e} in $\mathbb{k}Q^{\text{ab}}$ of the vertices e in Q_0 are a complete set of central orthogonal idempotents and $\bar{e}(\mathbb{k}Q^{\text{ab}})\bar{e}$ is isomorphic to $(e\mathbb{k}Qe)^{\text{ab}}$, and thus isomorphic to B_e . As a consequence, there is an isomorphism $\theta : \mathbb{k}Q^{\text{ab}} \rightarrow \bigoplus_{e \in Q_0} B_e$ such that $\theta(\pi(\alpha)) = X_\alpha$ for all $\alpha \in e\mathbb{k}Qe$ and $e \in Q_0$. Let g be the map $f^{\text{ab}} \circ \theta^{-1}$. The commutativity of the diagram

$$\begin{array}{ccccc} \mathbb{k}Q & \xrightarrow{\pi} & \mathbb{k}Q^{\text{ab}} & \xrightarrow{\theta} & \bigoplus_{e \in Q_0} B_e \\ \downarrow f & & \downarrow f^{\text{ab}} & \nearrow g & \\ \mathbb{k}Q/I & \longrightarrow & (\mathbb{k}Q/I)^{\text{ab}} & & \end{array}$$

implies that g is surjective and that its kernel is $(\theta \circ \pi)(I + J_{\mathbb{k}Q}) = (\theta \circ \pi)(I) = \bigoplus_{e \in Q_0} I_e$, and the lemma follows. \square

The following lemma is a well known result for finite dimensional algebras replacing Tor by Ext. Here we give the proof for completeness.

Lemma 3.2. *Let $B = \mathbb{k}Q/I$ be a monomial algebra. For each $e \in Q_0$ denote by T_e the simple B -module corresponding to e . If $e, e' \in Q_0$, then $\#eQ_1e' = \dim_{\mathbb{k}} \text{Tor}_1^B(T_e, T_{e'})$. Moreover, if Q has only one vertex e , then the dimension of $\text{Tor}_1^B(T_e, T_e)$ is*

$$\sup\{\dim_{\mathbb{k}} \text{Tor}_1^B(T_1, T_2) : T_1, T_2 \text{ are one dimensional } B\text{-modules}\}.$$

Proof. Let $e, e' \in Q_0$. The first terms of the minimal projective bimodule resolution of B are

$$\cdots \rightarrow B \otimes_E \mathbb{k}Q_1 \otimes_E B \rightarrow B \otimes_E B \rightarrow B \rightarrow 0,$$

where $E = \mathbb{k}Q_0$.

Applying the functor $T_e \otimes_B (-) \otimes_B T_{e'}$ to this resolution we obtain the following complex

$$\cdots \rightarrow T_e \otimes_{\mathbb{k}} \mathbb{k}eQ_1e' \otimes_{\mathbb{k}} T_{e'} \rightarrow T_e \otimes_{\mathbb{k}} T_{e'} \rightarrow 0,$$

whose homology is isomorphic to $\text{Tor}_{\bullet}^B(T_e, T_{e'})$. The minimality of the resolution implies that every arrow in the above complex is zero. As a consequence, $\text{Tor}_1^B(T_e, T_{e'}) \cong T_e \otimes_{\mathbb{k}} \mathbb{k}eQ_1e' \otimes_{\mathbb{k}} T_{e'} \cong e\mathbb{k}Q_1e'$, from where we deduce that the dimension of $\text{Tor}_1^B(T_e, T_{e'})$ is $\#eQ_1e'$. As for the second assertion, the same argument shows that if T_1, T_2 are one dimensional B -modules, then the homology of the complex

$$\cdots \rightarrow T_1 \otimes_{\mathbb{k}} \mathbb{k}Q_1 \otimes_{\mathbb{k}} T_2 \rightarrow T_1 \otimes_{\mathbb{k}} T_2 \rightarrow 0,$$

is isomorphic to $\text{Tor}_{\bullet}^B(T_1, T_2)$. It follows that $\dim_{\mathbb{k}} \text{Tor}_1^B(T_1, T_2) \leq \dim_{\mathbb{k}}(\mathbb{k}Q_1) = \#Q_1 = \dim_{\mathbb{k}} \text{Tor}_1^B(T_e, T_e)$, where e is the only vertex of Q . \square

Lemma 3.3. *Let $A = A(\alpha, 0, \gamma)$ and let T_1, T_2 be one dimensional A -modules.*

- (i) *If $\gamma \neq 0$ and $\alpha = 1$, then $\dim_{\mathbb{k}} \text{Tor}_1^A(T_1, T_2) = 0$.*
- (ii) *If $\gamma \neq 0$ and $\alpha \neq 1$, then $\dim_{\mathbb{k}} \text{Tor}_1^A(T_1, T_2) \leq 1$.*

(iii) If $\gamma = 0$, then $\dim_{\mathbb{k}} \operatorname{Tor}_1^A(T_1, T_2) \leq 2$ and $\dim_{\mathbb{k}} \operatorname{Tor}_1^A(\mathbb{k}, \mathbb{k}) = 2$. Moreover, if $\alpha \neq 1$ and $T_1 \neq \mathbb{k}$, then $\dim_{\mathbb{k}} \operatorname{Tor}_1^A(T_1, T_1) = 1$.

Proof. Let T_1, T_2 be one dimensional A -modules with bases $\{v_1\}$ and $\{v_2\}$, respectively. Let $\delta_1, \delta_2, \mu_1, \mu_2 \in \mathbb{k}$ be such that $d \cdot v_i = \delta_i v_i$ and $u \cdot v_i = \mu_i v_i$ for $i = 1, 2$. From the equalities $d^2 u - \alpha d u d - \gamma d = 0 = du^2 - \alpha u d u - \gamma u$ we deduce

$$\begin{aligned}\delta_i((1 - \alpha)\delta_i \mu_i - \gamma) &= 0, \\ \mu_i((1 - \alpha)\delta_i \mu_i - \gamma) &= 0,\end{aligned}\tag{3.1}$$

for $i = 1, 2$. Consider the following resolution of A as A -bimodule [4]

$$0 \rightarrow A \otimes_{\mathbb{k}} \Omega \otimes_{\mathbb{k}} A \xrightarrow{d_3} A \otimes_{\mathbb{k}} R \otimes_{\mathbb{k}} A \xrightarrow{d_2} A \otimes_{\mathbb{k}} V \otimes_{\mathbb{k}} A \xrightarrow{d_1} A \otimes_{\mathbb{k}} A \rightarrow 0,$$

where V, R and Ω are the subspaces of the free algebra $\mathbb{k}\langle d, u \rangle$ spanned, respectively, by the sets $\{d, u\}$, $\{d^2 u, du^2\}$ and $\{d^2 u^2\}$. The differentials are

$$\begin{aligned}d_1(1 \otimes v \otimes 1) &= v \otimes 1 - 1 \otimes v, \quad \text{for all } v \in V, \\ d_2(1 \otimes d^2 u \otimes 1) &= 1 \otimes d \otimes du + d \otimes d \otimes u + d^2 \otimes u \otimes 1 \\ &\quad - \alpha(1 \otimes d \otimes ud + d \otimes u \otimes d + du \otimes d \otimes 1) \\ &\quad - \beta(1 \otimes u \otimes d^2 + u \otimes d \otimes d + ud \otimes d \otimes 1) \\ &\quad - \gamma \otimes d \otimes 1, \\ d_2(1 \otimes du^2 \otimes 1) &= 1 \otimes d \otimes u^2 + d \otimes u \otimes u + du \otimes u \otimes 1 \\ &\quad - \alpha(1 \otimes u \otimes du + u \otimes d \otimes u + ud \otimes u \otimes 1) \\ &\quad - \beta(1 \otimes u \otimes ud + u \otimes u \otimes d + u^2 \otimes d \otimes 1) \\ &\quad - \gamma \otimes u \otimes 1,\end{aligned}$$

and

$$\begin{aligned}d_3(1 \otimes d^2 u^2 \otimes 1) &= d \otimes du^2 \otimes 1 + \beta \otimes du^2 \otimes d \\ &\quad - 1 \otimes d^2 u \otimes u - \beta u \otimes d^2 u \otimes 1.\end{aligned}\tag{3.2}$$

Applying the functor $T_1 \otimes_A (-) \otimes_A T_2$ to this resolution of A we obtain the following complex of \mathbb{k} -vector spaces whose homology is isomorphic to $\operatorname{Tor}_{\bullet}^A(T_1, T_2)$,

$$0 \longrightarrow \mathbb{k} \xrightarrow{f_2} \mathbb{k}^2 \xrightarrow{f_1} \mathbb{k}^2 \xrightarrow{f_0} \mathbb{k} \longrightarrow 0,$$

where

$$f_0 = \begin{pmatrix} \delta_2 - \delta_1 & \mu_2 - \mu_1 \end{pmatrix} \text{ and}$$

$$f_1 = \begin{pmatrix} (1 - \alpha)\delta_1\mu_1 + \delta_2(\mu_1 - \alpha\mu_2) - \gamma & \mu_1(\mu_1 - \alpha\mu_2) \\ \delta_2(\delta_2 - \alpha\delta_1) & (1 - \alpha)\delta_2\mu_2 + \mu_1(\delta_2 - \alpha\delta_1) - \gamma \end{pmatrix}.$$

The claims of the lemma follow from these formulas and Equation (3.1). \square

We now turn to the proof of Theorem 1.2. Suppose $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. Denote as usual $A(\alpha, \beta, \gamma)$ by A . Let $B = \mathbb{k}Q/I$ be a monomial algebra and suppose there exists an isomorphism of \mathbb{k} -algebras $\varphi : A \rightarrow B$. Since every down-up algebra has global dimension 3 [6], we deduce that $I \neq 0$ and so B is not a domain, thus we get $\beta = 0$.

Suppose $\gamma = 0$. In this case $\alpha \neq 0$ since we are assuming $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. Note that

$$A^{\text{ab}} = \mathbb{k}[d, u] / \langle (1 - \alpha)d^2u, (1 - \alpha)du^2 \rangle,$$

in particular it is connected and so is B^{ab} . By Lemma 3.1, the quiver Q has only one vertex e . Moreover, by Lemmas 3.2 and 3.3 we deduce

$$2 = \dim_{\mathbb{k}}(T_e, T_e) = \#Q_1,$$

thus, Q has exactly two arrows a and b . By Lemma 2.3 the quantum plane $\mathbb{k}_{\alpha}[x, y]$ is a quotient of A and so there exists an epimorphism $\varphi : B \rightarrow \mathbb{k}_{\alpha}[x, y]$. Since $\alpha \neq 0$, the quantum plane $\mathbb{k}_{\alpha}[x, y]$ is a domain. Given a non zero path p in I , $\varphi(p) = 0$, implying that either $\varphi(a)$ or $\varphi(b)$ is zero. As a consequence, the quantum plane is generated as algebra by one variable, which is a contradiction.

Now suppose $\gamma \neq 0$. If $\alpha = 1$, then $\text{Tor}_1^A(T_1, T_2) = 0$ for every pair of one dimensional A -modules T_1, T_2 , from Lemma 3.3. Since $A \cong B$, this is also true for B and one dimensional B -modules. By Lemma 3.2, the quiver Q has no arrows. This is impossible, and so $\alpha \neq 1$. The same lemmas imply in this case that there is at most one arrow between each pair of vertices in Q . In particular, there is at most one element in eQ_1e for every vertex e .

Define $V = \{e \in Q_0 : \#eQ_1e = 1\}$ and for every $e \in V$, denote a_e the unique element in eQ_1e . Since A , and thus B , is of global dimension 3, Bardzell's resolution [1] of B is of finite length. So $a_e^n \notin I$ for all $n \in \mathbb{N}$. This implies $B_e = \mathbb{k}[X]$ and $I_e = 0$ for all $e \in V$, and $B_e = \mathbb{k}$ for all $e \notin V$. By Lemma 3.1,

$$B^{\text{ab}} \cong \bigoplus_{e \notin V} \mathbb{k} \oplus \bigoplus_{e \in V} \mathbb{k}[X].$$

In particular, its group of units is contained in the finite dimensional vector space $\mathbb{k}^{\#Q_0}$. On the other hand, the fact that the ideals $\langle d, u \rangle$ and $\langle (1 - \alpha)du - \gamma \rangle$ in $\mathbb{k}[d, u]$ are coprime implies

$$A^{\text{ab}} \cong \mathbb{k} \oplus \frac{\mathbb{k}[d, u]}{\langle (1 - \alpha)du - \gamma \rangle}.$$

The group of units of this algebra is contained in no finite dimensional space. Since B^{ab} is isomorphic to A^{ab} , this is a contradiction and we conclude the proof of Theorem 1.2.

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